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The transmission of deep-water waves across a vortex sheet

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The effect on an obliquely incident surface wave of a vortex sheet separating two uniform currents is considered. It is shown that the amplitude of the transmitted wave as a function of the angle of incidence and current strength is very close to that obtained by Longuet-Higgins & Stewart on the assumption of small smooth changes in current velocity. The difference is accounted for by a small amount of reflexion of the wave by the vortex sheet. It is suggested that, in the intermediate range where the change in current velocity over a wavelength is comparable with the wave velocity, the wave amplitude of the transmitted wave lies between the curves of Longuet-Higgins & Stewart and those found here.

1. Introduction

The problem which is considered in this paper is the propagation of a plane surface wave across a vortex sheet separating two regions of fluid having different uniform velocities U_1 and U_2. The problem is complicated by the free-surface condition which has to be satisfied. If the free surface could be replaced by a rigid lid, then the problem would reduce to the classical Kelvin–Helmholtz stability problem discussed in Lamb (1932, §232).

The question of the stability of the vortex sheet in the present problem is a difficult one. The usual approach is to consider an appropriate initial-value problem. Miles (1958) has done this for the problem of the transmission of sound waves across a shear layer. He found that, because of sound radiation, fewer instabilities occurred at supersonic speeds. Jones & Morgan (1972) have considered the corresponding problem in which one medium contains an acoustic line source parallel to the interface. They solved the initial-value problem and showed that, in order to satisfy the causality condition, it was necessary to introduce into the harmonic solution an additional term which represented a disturbance which grew exponentially downstream but which decayed exponentially in the direction perpendicular to the vortex sheet. Such a procedure does not appear possible in the present case because of the complications introduced by the free-surface boundary condition, and even the harmonic problem has not been solved explicitly. It is the purpose of this paper to present an approximate solution for the harmonic case.

Although the presence of the free surface precludes a solution with the simple structure of the classical Kelvin–Helmholtz instability it seems likely that instabilities must occur in the sheet at depths at which the influence of the free surface is negligible. However it is possible, by analogy with the acoustic case, that surface wave radiation from the vortex sheet could have a stabilizing influence.

In the present work we shall consider only the simplest problem of a plane wave obliquely incident upon the vortex sheet and we shall assume that all disturbances are time-harmonic and small enough for the linearized theory to hold, so that instabilities are excluded.

On the experimental side, Savitsky's (1970) investigation of the interactions of waves and turbulence showed that even small mean currents had a greater effect on an incident wave than the turbulence which he generated. Thus it seems likely that any turbulence generated by a shear layer will have a smaller influence than the mean velocity gradients in the flow. Hence the present idealized model of the flow may be a good representation of some physical situations.

Previous work on the water-wave case includes that of Johnson (1947), who determined the changes in wavelength and direction of the waves from largely kinematic considerations, but who derived expressions for the change in wave height from energy considerations based only the far field with no attempt to match the near field across the interface. More recently, Maruo & Hayasaki (1972), who were interested in the propagation of waves through a ship's wake, have used a Green's function technique to solve the full problem including the matching of the near field across the vortex sheet. Unfortunately one of the boundary conditions across the sheet (their equation 2) is incorrect and this invalidates their results. The method used here to formulate the problem involves the use of complete eigenfunction expansions and is equivalent to the Green's function approach of Maruo & Hayasaki. However, whereas they attempted a full numerical solution to the resulting integral equations, a simpler but accurate approximate method is used here.

The main aim of the present work is to compare the results with these obtained by Longuet-Higgins & Stewart (1961, hereafter denoted by I). They used the concept of radiation stress introduced in a previous paper (Longuet-Higgins & Stewart 1960) to consider the changes in wave amplitude of a surface wave as it propagated across a region of current shear. They used a ray-theory approximation which was valid provided that the current varied slowly and smoothly on a scale of several wavelengths. In their case the amount of wave energy reflected was exponentially small and a nonlinear coupling produced a transfer of energy between the waves and the current. The model used here can be regarded as the first approximation in a perturbation scheme based on the contrary assumption that the current varies rapidly on a scale much shorter than a wavelength.

The results are contained in figures 1 and 2, where it can be seen that the sharp change in current speed results in a surprisingly small amount of reflexion $|R_1|$ of the incident wave. Figure 1 shows how close the curves of the transmission coefficients $|T_1|$ *vs.* current speed are to the results in I. This suggests that in the intermediate range, where the current speed varies appreciably over a wavelength and where neither theory is applicable, the curves of transmitted wave height lie between those shown in figure 1.

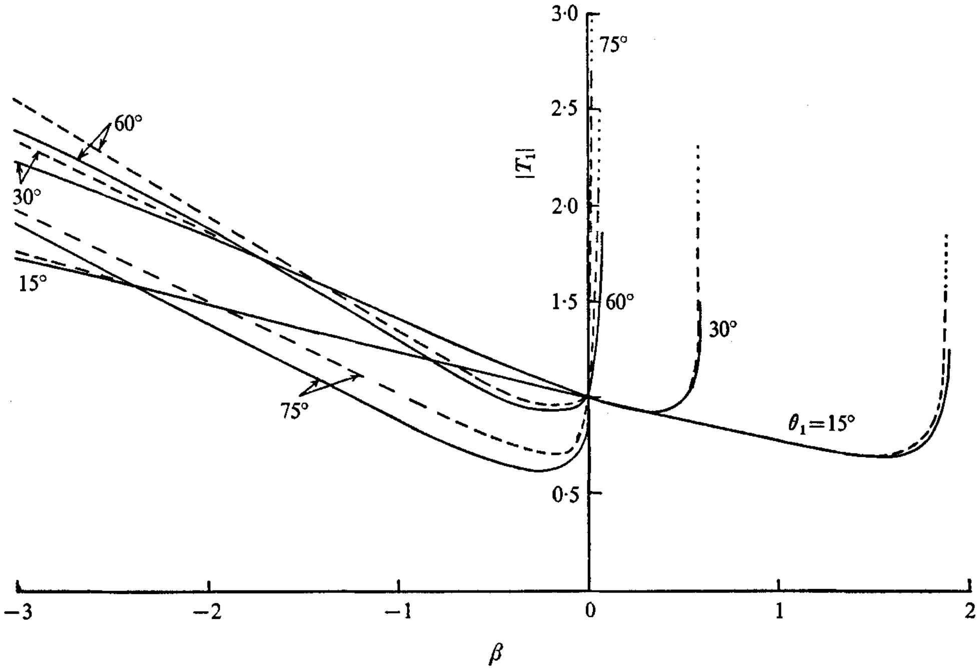

FIGURE 1. A comparison of the modulus $|T_1|$ of the transmission coefficient derived from the approximation based on the velocity formulation (solid lines) with that obtained by Longuet-Higgins & Stewart (dashed lines).

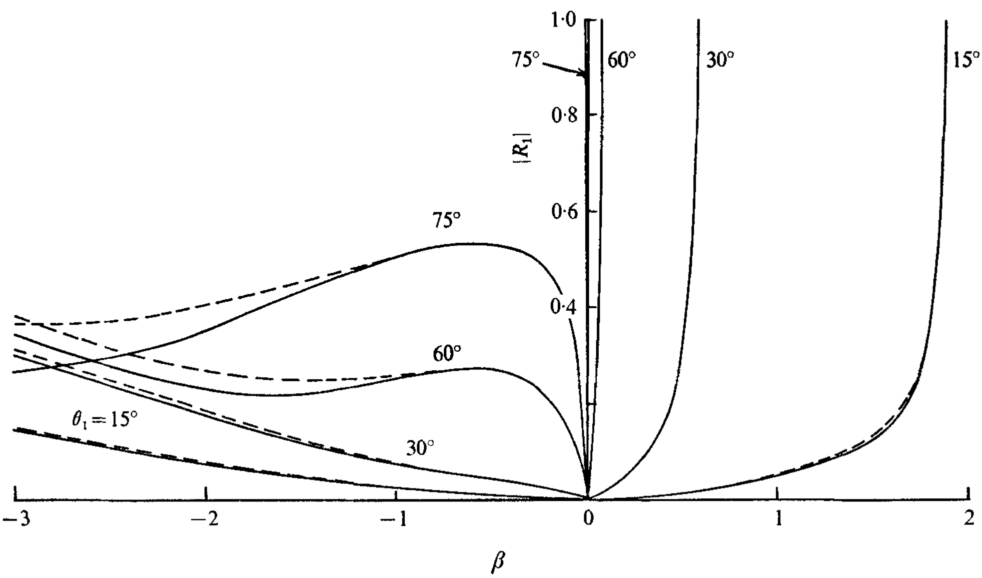

FIGURE 2. A comparison of the modulus $|R_1|$ of the reflexion coefficient derived from the approximation based on the velocity formulation (solid lines) with that based on the potential formulation (dashed lines).

A simpler problem having an exact solution for $|R_1|$ and $|T_1|$ is considered in the final section and a comparison is made between the present approximation and the simpler exact results derived by Keller & Weitz (1953). The agreement is shown to be good over a wide range of the parameters of the problem.

2. Formulation

Axes are chosen such that the x, z plane is the undisturbed free surface and y points vertically downwards. It is assumed that uniform flows with constant velocities U_1 and U_2, in the z direction, exist in $x \gtrless 0$, respectively. The final solution will depend only on the relative velocity $U_1 - U_2$. However, there is some advantage to be gained in terms of symmetry and notational convenience by retaining the separate indentities of U_1 and U_2 at this stage. The fluid is assumed to be incompressible and irrotational so that a velocity potential $\Phi(x, y, z, t)$ satisfying Laplace's equation in the fluid exists. The pressure in the fluid is given by Bernoulli's equation

$$p(x, y, z, t) + \rho\{\partial\Phi/\partial t - gy + \tfrac{1}{2}(\nabla\Phi)^2\} = F(t),$$

where ρ is the fluid density, g the acceleration due to gravity and $F(t)$ a function of time only.

Let

$$\Phi = \begin{cases} U_1 z + \phi_1(x, y, z, t), & x > 0, \\ U_2 z + \phi_2(x, y, z, t), & x < 0, \end{cases} \tag{2.1}$$

where ϕ_1 and ϕ_2 are perturbation potentials, and let the perturbed free surface have the equation $y = \eta(x, z, t)$. Then the continuity of pressure across the free surface gives, to first order in ϕ_m and η,

$$\frac{\partial \phi_m}{\partial t} - g\eta + U_m \frac{\partial \phi_m}{\partial z} = 0, \quad y = 0 \quad (m = 1, 2), \tag{2.2}$$

whilst the condition that a fluid particle remains on the free surface, also to first order in ϕ_m and η, is

$$\frac{\partial \phi_m}{\partial y} = \frac{\partial \eta}{\partial t} + U_m \frac{\partial \eta}{\partial z}, \quad y = 0 \quad (m = 1, 2). \tag{2.3}$$

Elimination of η between (2.2) and (2.3) gives

$$\left(\frac{\partial}{\partial t} + U_m \frac{\partial}{\partial z}\right)^2 \phi_m = g \frac{\partial \phi_m}{\partial y}, \quad y = 0 \quad (m = 1, 2). \tag{2.4}$$

In the perturbed motion, the equation of the interface separating the uniform flows is assumed to be $x = \xi(y, z, t)$. Equations relating ϕ_1 and ϕ_2 across the vortex sheet are obtained by applying the condition of continuity of pressure across the sheet and the condition that a fluid particle remains on the sheet. We find that

$$\frac{\partial \phi_1}{\partial t} + U_1 \frac{\partial \phi_1}{\partial z} = \frac{\partial \phi_2}{\partial t} + U_2 \frac{\partial \phi_2}{\partial z}, \quad x = 0, \tag{2.5}$$

$$\frac{\partial \phi_m}{\partial x} = \frac{\partial \xi}{\partial t} + U_m \frac{\partial \xi}{\partial z}, \quad x = 0 \quad (m = 1, 2), \tag{2.6}$$

both equations being satisfied to first order in ϕ_m and ξ. We shall not consider the question of Kelvin–Helmholtz instabilities occurring in the vortex sheet. It will be assumed that throughout the motion the perturbations in the sheet are such that (2.5) and (2.6) continue to hold. Elimination of ξ from (2.6) gives

$$\left(\frac{\partial}{\partial t}+U_1\frac{\partial}{\partial z}\right)\frac{\partial\phi_2}{\partial x}=\left(\frac{\partial}{\partial t}+U_2\frac{\partial}{\partial z}\right)\frac{\partial\phi_1}{\partial x},\quad x=0. \tag{2.7}$$

In addition the perturbation potentials satisfy Laplace's equation throughout the fluid.

We shall assume that a sinusoidal wave motion exists in the sheet, having frequency ω and wavenumber p. It follows from (2.6) that we must have

$$\phi_m(x,y,z,t)\equiv\phi_m(x,y)\exp\{ipz-i\omega t\}\quad(m=1,2), \tag{2.8}$$

where the real part is to be taken finally. Then $\phi_m(x,y)$ satisfies

$$(\nabla^2-p^2)\phi_m=0\quad(m=1,2,\quad\nabla^2\equiv\partial^2/\partial x^2+\partial^2/\partial y^2) \tag{2.9}$$

throughout the fluid,

$$K_m\phi_m+\partial\phi_m/\partial y=0,\quad y=0,\quad -\infty<x<\infty\quad(m=1,2), \tag{2.10}$$

$$\alpha_1\phi_1=\alpha_2\phi_2,\quad x=0,\quad y>0, \tag{2.11}$$

$$\alpha_1^{-1}\partial\phi_1/\partial x=\alpha_2^{-1}\partial\phi_2/\partial x,\quad x=0,\quad y>0, \tag{2.12}$$

where

$$\alpha_m=K_m^{\frac{1}{2}}=|\omega-pU_m|/g^{\frac{1}{2}}. \tag{2.13}$$

We shall be concerned with the propagation of an obliquely incident wave across the vortex sheet. Let the angle between the crests of the wave and the z axis in $x\gtrless 0$ be θ_1 and θ_2 respectively. In what follows we use a notation similar to that of Miles (1967). We can construct complete eigenfunction expansions in $x\gtrless 0$ which satisfy (2.9) and (2.10). Thus

$$\phi_m(x,y)=\operatorname{sgn}x\left\{(A_m e^{-il_m|x|}+B_m e^{il_m|x|})\chi_m(y)+\int_0^\infty C_m(k)\,e^{-k'|x|}\psi_m(y,k)\,dk\right\}\quad(m=1,2), \tag{2.14}$$

where $l_m=K_m\cos\theta_m$ and $k'=(k^2+p^2)^{\frac{1}{2}}$, so that $p=K_m\sin\theta_m$ $(m=1,2)$. The functions

$$\chi_m(y)=(2K_m)^{\frac{1}{2}}\exp(-K_m y)$$

and

$$\psi_m(y,k)=(2/\pi)^{\frac{1}{2}}(k^2+K_m^2)^{-\frac{1}{2}}(K_m\sin ky-k\cos ky)$$

satisfy the orthonormality conditions

$$\left.\begin{aligned}\int_0^\infty\chi_m^2(y)dy&=1,\\ \int_0^\infty\psi_m(y,k)\chi_m(y)dy&=0,\\ \int_0^\infty\psi_m(y,k_1)\psi_m(y,k_2)\,dy&=\delta(k_1-k_2).\end{aligned}\right\} \tag{2.15}$$

Now from (2.14)

$$[\partial\phi_m/\partial x]_{x=0}\equiv\mathscr{U}_m(y)=-il_m(A_m-B_m)\chi_m(y)-\int_0^\infty k'C_m(k)\psi_m(y,k)\,dk\quad(m=1,2),$$

so that from (2.15)

$$\left.\begin{aligned} -il_m(A_m - B_m) &= \int_0^\infty \mathscr{U}_m(t)\,\chi_m(t)\,dt, \\ -k'C_m(k) &= \int_0^\infty \mathscr{U}_m(t)\,\psi_m(t,k)\,dt. \end{aligned}\right\} \tag{2.16}$$

An integral equation for $\mathscr{U}_m(y)$ is now obtained by applying conditions (2.11) and (2.12)–(2.14), using (2.16). Thus

$$\sum_{m=1}^{2} \alpha_m(A_m + B_m)\,\chi_m(y) = \int_0^\infty \mathscr{U}(t)\,G(y,t)\,dt, \tag{2.17}$$

where

$$\alpha_1^{-1}\mathscr{U}_1(y) = \alpha_2^{-1}\mathscr{U}_2(y) \equiv \mathscr{U}(y)$$

and

$$G(y,t) = \sum_{m=1}^{2} \int_0^\infty \alpha_m^2 \psi_m(y,k)\,\psi_m(t,k)\,k'^{-1}\,dk. \tag{2.18}$$

At this stage it is convenient to introduce matrix notation. We define

$$\mathbf{A}^{\mathrm{T}} = (A_1, A_2), \quad \mathbf{B}^{\mathrm{T}} = (B_1, B_2), \quad \mathbf{d}^{\mathrm{T}} = (d_1, d_2) = (\alpha_1\chi_1, \alpha_2\chi_2)$$

and

$$\mathbf{l} = \operatorname{diag}\{l_1, l_2\}.$$

Then (2.16) and (2.17) become

$$-i\mathbf{l}(\mathbf{A} - \mathbf{B}) = \int_0^\infty \mathbf{d}\,\mathscr{U}(t)\,dt, \tag{2.19}$$

$$\mathbf{d}^{\mathrm{T}}(\mathbf{A} + \mathbf{B}) = \int_0^\infty \mathscr{U}(t)\,G(y,t)\,dt. \tag{2.20}$$

Following Miles (1967) we define the scattering matrix $\mathbf{S}$ by

$$\mathbf{l}(\mathbf{A} - \mathbf{B}) = i\mathbf{S}(\mathbf{A} + \mathbf{B}). \tag{2.21}$$

Then the waves at either infinity are related by

$$\mathbf{B} = \mathbf{T}\mathbf{A}, \tag{2.22}$$

where

$$\mathbf{T} = (\mathbf{l} + i\mathbf{S})^{-1}(\mathbf{l} - i\mathbf{S}). \tag{2.23}$$

If we now introduce a normalized velocity $\mathbf{u}(y)$ defined by

$$\mathscr{U}(y) = \mathbf{u}^{\mathrm{T}}(y)\,(\mathbf{A} + \mathbf{B}), \tag{2.24}$$

we obtain

$$d_m(y) = \int_0^\infty u_m(t)\,G(y,t)\,dt, \quad 0 < y < \infty \quad (m = 1, 2), \tag{2.25}$$

$$S_{mn} = \int_0^\infty d_m(t)\,u_n(t)\,dt \quad (m, n \in \{1, 2\}). \tag{2.26}$$

The problem has been reduced to the solution of the two singular integral equations given by (2.25). Once these have been solved for $\mathbf{u}(y)$, the matrix $\mathbf{S}$ may be determined from (2.26) and the wave amplitudes at either infinity may be related to the amplitude of the incident wave using (2.2), (2.8), (2.22) and (2.23). Thus the wave amplitude at either infinity is given by

$$\operatorname{Re}\{-ig^{-\frac{1}{2}}\alpha_m\phi_m(x,0)\exp[i(pz - \omega t)]\} \quad (m = 1, 2). \tag{2.27}$$

It is possible to obtain useful information about the solution before actually solving for $\mathbf{u}(y)$. We shall define the complex transmission coefficient T_1 (T_2) to be the ratio of the complex amplitude (2.27) of the transmitted wave at $x = -\infty$ ($+\infty$) to that of the incident wave at $x = +\infty$ ($-\infty$). The complex reflexion coefficients R_1 and R_2 will be defined correspondingly. Then it follows from (2.14), (2.22) and (2.27) that with $A_2 = 0$

$$T_1 = -\mu^{-1}T_{21}, \quad R_1 = T_{11} \tag{2.28}$$

and with $A_1 = 0$

$$T_2 = -\mu T_{12}, \quad R_2 = T_{22}, \tag{2.29}$$

where $\mu = K_1/K_2$. Also from (2.23) we can derive the results

$$\arg T_1 = \arg T_2, \quad |R_1| = |R_2|, \quad l_1 T_{12} = l_2 T_{21},$$

giving

$$\cos\theta_2 T_1 = \cos\theta_1 T_2,$$

and also

$$|R_1|^2 + (\sin 2\theta_2/\sin 2\theta_1)|T_1|^2 = 1. \tag{2.30}$$

This last result can also be derived from a simple application of Green's theorem to ϕ_m and its complex conjugate.

Equation (2.30) merits further discussion since it is not equivalent to a straightforward energy balance across the vortex sheet. It has been pointed out by Whitham (1962) that in a linearized problem of the type considered here, where the currents are prescribed beforehand to first order, any second-order mean currents induced by the waves may introduce spurious terms in an energy equation based on physical considerations. The correct interpretation of (2.30) is that of conservation of *wave action*, defined as energy flux divided by intrinsic frequency, the latter being the frequency measured by an observer travelling with the flow.

Although conservation of wave action is more usually associated with problems involving slowly varying wave trains, support for the above result in the present problem, where an abrupt change in current speed occurs, can be found in the work of Hayes (1970). He showed that, for a wide class of systems having a Lagrangian density and with periodic solutions, wave action was conserved with no requirement that a parameter be small.

The result in I, equation (8.16), for the amplification of short waves as they traverse a slowly varying current, can also be interpreted as conservation of wave action and the result is identical to (2.30) if R_1 is put equal to zero.

3. An approximate solution

There can be little hope of inverting (2.25) explicitly. Here we shall employ a two-term Galerkin approximation using exponentials appropriate to either side of the vortex sheet. Thus we shall assume that

$$u_m(t) = \sum_{s=1}^{2} c_{ms} d_s(t) \quad (m = 1, 2), \tag{3.1}$$

where the c_{ms} are constants.

We substitute (3.1) into (2.25), multiply by $d_n(y)$ and integrate from 0 to ∞. We find, in matrix notation, that $\mathbf{F} = \mathbf{CE}$, where $\mathbf{F} = \{f_{mn}\}$, $\mathbf{E} = \{e_{mn}\}$, $\mathbf{C} = \{c_{mn}\}$ and

$$f_{mn} = f_{nm} = \int_0^\infty d_m(y)\,d_n(y)\,dy \qquad (m = 1, 2). \tag{3.2}$$

$$e_{mn} = e_{nm} = \int_0^\infty \int_0^\infty d_m(t)\,d_n(y)\,G(y,t)\,dt\,dy \tag{3.3}$$

Furthermore the approximation to $\mathbf{S}$ from (2.26) is $\mathbf{S} \simeq \mathbf{S}^v = \mathbf{CF}$, whence

$$\mathbf{S}^v = \mathbf{FE}^{-1}\mathbf{F} \tag{3.4}$$

and $\mathbf{S}^v$ is symmetric.

An alternative formulation and approximate solution to the problem are possible using the potential across $x = 0$ instead of the horizontal velocity as the unknown. Analysis similar to that leading to (2.25) and (2.26) gives $\mathbf{S} \simeq \mathbf{S}^p$, where

$$\mathbf{S}^p = \mathbf{L}^{-1}\mathbf{K}\mathbf{L}^{-1} \tag{3.5}$$

and
$$\mathbf{L} = \{l_{mn}\}, \quad \mathbf{K} = \{k_{mn}\},$$

$$l_{mn} = l_{nm} = \int_0^\infty h_m(y)\,h_n(y)\,dy, \tag{3.6}$$

$$k_{mn} = k_{nm} = \int_0^\infty \int_0^\infty h_m(y)\,h_n(t)\,H(y,t)\,dt\,dy, \tag{3.7}$$

with
$$H(y,t) = \sum_{m=1}^{2} \int_0^\infty k'\alpha_m^{-2}\psi_m(t)\,\psi_m(y)\,dk$$

and
$$\mathbf{h}^{\mathrm{T}}(y) = (h_1, h_2) = (\alpha_1^{-1}\chi_1, -\alpha_2^{-2}\chi_2).$$

Once $\mathbf{S}^v$ and $\mathbf{S}^p$ have been computed we can relate the far-field properties of the solution through (2.22) and (2.23).

The integrals occurring in (3.2), (3.3), (3.6) and (3.7) may all be worked out explicitly, and from (3.4) and (3.5), without too much trouble, we find for the velocity approximation

$$S_{11}^v = A^v\mu + B^vC^2, \quad S_{22}^v = A^vC^2 + B^v/\mu,$$
$$S_{12}^v = S_{21}^v = (A^v\mu + B^v)C/\mu^{\frac{1}{2}},$$

where
$$A^v = \tfrac{1}{4}\pi K_2\mu(1+\mu)^2/\{(M_1 - M_2)\mu^2 + (1-\mu^2)H_1\}, \tag{3.8a}$$

$$B^v = \tfrac{1}{4}\pi K_2(1+\mu)^2/\{M_2 - M_1) - (1-\mu^2)H_2\}, \tag{3.8b}$$

$$C = 2\mu^{\frac{1}{2}}/(1+\mu), \quad \mu = K_1/K_2$$

and
$$M_n = \sec\theta_n \log\cot\tfrac{1}{2}\theta_n, \quad H_n = \tfrac{1}{2}(1 - M_n\sin^2\theta_n) \quad (n = 1, 2).$$

In the potential approximation we find that the S_{mn}^p can be expressed in a similar way with A^v and B^v replaced by A^p and B^p, where

$$A^p = (4/\pi)K_2\mu(1+\mu)^2(1-\mu)^{-4}\{\cos^2\theta_2 M_2 - \cos^2\theta_1 M_1 - (1-\mu^2)L_1\}, \tag{3.9a}$$

$$B^p = (4/\pi)K_2(1+\mu)^2(1-\mu)^{-4}\{\mu^2(\cos^2\theta_1 M_1 - \cos^2\theta_2 M_2) + (1-\mu^2)L_2\}, \tag{3.9b}$$

$$L_n = \tfrac{1}{2}(1 + M_n\sin^2\theta_n) \quad (n = 1, 2).$$

It had been hoped that the expressions for S^p_{mn} and S^v_{mn} could be shown to be upper and lower bounds respectively to the true value of S_{mn}, but this is only possible for $m = n$ (see Jones (1964, p. 271) for a proof in this case). In any event complementary bounds on S_{mn} do not give corresponding bounds on T_{mn} because of the complicated relation between **S** and **T**.

In computing reflexion and transmission coefficients, we shall, without loss of generality, assume that the incident wave is propagating from $x = +\infty$ in an otherwise stationary fluid. This is equivalent to applying a Galilean transformation of co-ordinates in z and t. Then $U_1 = 0$ and there can be no incoming wave at $x = -\infty$, so that $A_2 = 0$.

We find from (2.13) that

$$\sin\theta_2 = \sin\theta_1/(1-\beta\sin\theta_1)^2, \tag{3.10}$$

where $\beta = U_2/c_1$ and the relation $c_m = \omega/K_m$ has been used, c_1 (c_2) being the phase velocity of waves in $x > 0$ (< 0). For a given θ_1 in the interval $(0, \frac{1}{2}\pi)$ equation (3.10) has a solution for θ_2 for all β lying outside the range

$$[1-(\sin\theta_1)^{\frac{1}{2}}]/\sin\theta_1 < \beta < [1+(\sin\theta_1)^{\frac{1}{2}}]/\sin\theta_1. \qquad (3.11)\dagger$$

Furthermore, to each solution θ_2 correspond two values of β, both positive. So from (3.10) the effect of an opposing current ($\beta < 0$) on an obliquely incident wave is to reduce the angle between the crests and the positive z axis. For positive currents ($\beta > 0$) the angle increases up to $\frac{1}{2}\pi$ when β reaches the value

$$[1-(\sin\theta_1)^{\frac{1}{2}}]/\sin\theta_1.$$

For larger values of β total reflexion occurs and no wave propagates across the current until β reaches the value $[1+(\sin\theta_1)^{\frac{1}{2}}]/\sin\theta_1$. For β greater than this value transmission occurs once more and the angle increases from $\frac{1}{2}\pi$ to π as β increases indefinitely. This can be explained physically as follows. Up to a certain value of β it is possible to match the frequency and phase speed of a wave incident upon a weak positive current with a longer wave propagating down the current. As the strength of the current increases, matching becomes impossible and total reflexion occurs. For very strong positive currents it is possible to match the incident wave with a wave which is trying to propagate *up* the current but which is being swept back by the current, the sense of its phase speed thus being reversed. It is remarkable that for $\beta = 2/\sin\theta_1$ the incident wave alone provides a complete solution to the problem.‡ Once θ_2 (and hence μ from the relation $\mu\sin\theta_1 = \sin\theta_2$) has been determined from (3.10), the transmission and reflexion coefficients T_1 and R_1 may be computed from (2.23), (2.28) and the approximate expressions S^v and S^p. It also follows from the above discussion that curves of $|R_1|$ and $|T_1|$ *vs.* β are necessarily symmetric about the line $\beta = \operatorname{cosec}\theta_1$.

† The left-hand half of this inequality was noted by Longuet-Higgins & Stewart (1961, equation (8.10)), but they appear to have overlooked the possibility of further transmission for larger values of β.

‡ I am indebted to Dr D. H. Peregrine for this observation.

4. Discussion of results

Figure 1 shows how the ratio of the amplitudes of the transmitted and incident waves obtained by computing $|T_1|$ using the approximation based on the velocity formulation varies with the strength β $(= U_2/c_1)$ of the current for differing angles of incidence θ_1 of the incoming wave. The corresponding curves derived from the approximation based on the potential formulation are not shown as they coincide with the curves shown over almost the entire range of β covered. The difference between the two approximate values for $|T_1|$ increases with increasing θ_1 and $|\beta|$. Thus in the range $-3 < \beta < 2$ the greatest difference occurs at $\beta = -3$, where we find a difference of 0·04 % of the smaller value for $\theta_1 = 15°$, increasing to 3·2 % for $\theta_1 = 75°$. This near coincidence of the two approximations to $|T_1|$ makes it reasonable to suppose that the true curves for $|T_1|$ coincide with the curves shown within the limits given above.

Also shown in figure 1 are the corresponding curves obtained in I on the assumption that changes in current velocity over a wavelength are small compared with the wave velocity. In I the exact result $|T_1| = (\sin 2\theta_1/\sin 2\theta_2)^{\frac{1}{2}}$ was derived, with no reflexion of the incident wave, there being a transfer of energy between the waves and the current proportional to the radiation stress tensor. The concept of radiation stress was introduced in a previous paper by the same authors (Longuet-Higgins & Stewart 1960). In the present work it has been assumed that the change in current velocity from U_1 to U_2 occurs over a distance which is small compared with a wavelength, giving rise in the limit to a vortex sheet along $x = 0, y > 0$.

The most remarkable feature of the curves is the similarity, both qualitative and quantitative, between the results of I and the present work, especially in view of the marked difference in the assumptions made in deriving the mathematical models used in each case. It can be seen that the present amount of wave transmission is always slightly less than that given by the results of I. This is due to the small amount of wave energy reflected by the current; no reflexion of the wave occurred in I. The only other difference in the two sets of curves occurs near the critical value $\beta = \beta_{\text{crit}} = [1-(\sin\theta_1)^{\frac{1}{2}}]/\sin\theta_1$. In I the theory predicts $|T_1| \to \infty$ as $\beta \to \beta_{\text{crit}}$ whereas it can be shown that the present velocity approximation predicts that $|T_1| \to 1+\sin\theta_1$ as $\beta \to \beta_{\text{crit}}$. The present potential approximation gives a limit for $|T_1|$ which is also finite but different and more complicated. It is not clear what the true value for $|T_1|$ is in this limiting case, which appears to be a non-trivial problem in its own right.

In figure 2 curves of $|R_1|$ *vs.* β are shown for a range of values of θ_1. Here both the velocity and potential approximations are given and we see that the discrepancy between them is magnified for large θ_1 and $|\beta|$. This can be explained from (2.30) as follows. In an obvious notation, if

$$|T_1^v| = (1+\epsilon)|T_1^p| \quad \text{and} \quad |R_1^v| = (1+\delta)|R_1^p|,$$

then

$$\delta = -\frac{\sin 2\theta_2}{\sin 2\theta_1}\frac{|T_1^p|^2}{|R_1^p|^2}\epsilon \quad \text{approximately.} \tag{4.1}$$

Since $|R_1^p|^2$ is small, (4.1) shows how a small ϵ can lead to a large δ. Even so, in the range $|\beta| < 1$, which is perhaps that of greatest physical interest, the curves are very close together and it may be reasonably assumed that the approximations are very close to the true values. It is noticeable that for smaller incidence angles the amount of reflexion decreases monotonically to zero as the strength of the opposing current ($\beta < 0$) diminishes to zero. For larger values of θ_1 the reflexion first increases to a maximum value before falling to zero at $\beta = 0$. For example for a wave whose crests make an angle of 75° with the positive direction of the current, a maximum of about 25 % of the wave energy is reflected by a negative current whose velocity is about 60 % of the incident wave velocity. For positive currents ($\beta > 0$) the amount of reflexion increases sharply until the critical value β_{crit} is reached, where total reflexion occurs. As has already been mentioned, in both figures 1 and 2 the curves are symmetrical about the line $\beta = \operatorname{cosec}\theta_1$.

5. A simpler problem

It is possible to make a partial check on the accuracy of the approximations by comparing the results obtained for a simpler problem with the known exact solution for that problem. If in (2.11) and (2.12) we put $\alpha_1 = \alpha_2$, so that the perturbed potential and horizontal velocity in the x direction are continuous across $x = 0$, whilst at the same time keeping K_1 different from K_2, we obtain the boundary-value problem solved by Weitz & Keller (1950). Unlike the present problem, this simpler problem may be solved exactly using the Weiner–Hopf technique. Furthermore, it was remarked by Keller & Weitz (1953) in a subsequent paper that a simple expression exists for $|R_1|$, namely,

$$|R_1| = \left|\frac{\cos\theta_2 - \mu\cos\theta_1}{\cos\theta_2 + \mu\cos\theta_1}\right|, \tag{5.1}$$

where $\mu = K_1/K_2 = \sin\theta_2/\sin\theta_1$ as before. Keller & Weitz were concerned with the propagation of an obliquely incident wave into an ice field in the form of small non-interacting floating masses. Then for $\mu < 1$ the wave propagates into the ice field whilst for $\mu > 1$ the wave propagates out of the ice field and total reflexion occurs for $\mu > \mu_{\text{crit}} = \operatorname{cosec}\theta_1$. The result (5.1) can be compared with an approximate solution for the ice problem obtained by putting $\alpha_1 = \alpha_2 = 1$ in the expressions for $|R_1|$ derived for the shear problem. In turns out that the only modification required is the replacement of μ in the expressions for $S_{mn}^{v,p}$ by unity. The case of normal incidence, which is possible in the ice problem, requires special attention. It is found that if $\theta_1 = 0$ then $\theta_2 = 0$ and the expressions in curly brackets in (3.8a) and (3.9a) each become $2\mu^2\log\mu + 1 - \mu^2$, whilst the expressions in curly brackets in (3.8b) and (3.9a) each become $\mu^2 - \log\mu - 1$. Some care is needed in deriving the transmission coefficient in the ice problem, since the wave amplitude in the ice is given by

$$\operatorname{Re}\{(-iK_2/\omega)\,\phi_2(x, 0)\exp i(pz - \omega t)\}$$

	μ	$\lvert T_1\rvert$ Velocity approximation	Potential approximation	Exact value from (5.2)	$\lvert R_1\rvert$ Velocity approximation	Potential approximation	Exact value from (5.1)
$\theta_1 = 0$	0·1	6·0387	6·0386	5·7462	0·7966	0·7966	0·8182
	0·5	1·8862	1·8862	1·8853	0·3319	0·3319	0·3333
	0·9	1·1095	1·1095	1·1095	0·0526	0·0526	0·0526
	2·0	0·4716	0·4716	0·4714	0·3320	0·3320	0·3333
	3·0	0·2895	0·2895	0·2887	0·4955	0·4955	0·5000
	4·0	0·2015	0·2015	0·2000	0·5920	0·5920	0·6000
$\theta_1 = 30°$	0·1	5·2288	5·2938	5·0465	0·8274	0·8227	0·8404
	0·5	1·7485	1·7488	1·7481	0·3814	0·3810	0·3820
	0·9	1·0917	1·0917	1·0917	0·0679	0·0679	0·0679
	2·0†	0·7072	0·7075	0·7071	1·0000	1·0000	1·0000
$\theta_1 = 60°$	0·1	3·1094	3·1666	3·0225	0·8985	0·8945	0·9044
	0·5	1·2285	1·2288	1·2283	0·5655	0·5652	0·5657
	1·155†	1·6111	1·6111	1·6111	1·0000	1·0000	1·0000

† $\mu = \mu_{\text{crit}}$.

TABLE 1

and not, as in the shear problem, by (2.27). Thus we find that for the ice problem

$$|T_1| = 2\cos\theta_1/(\cos\theta_2 + \mu\cos\theta_1)\mu^{\frac{1}{2}} \tag{5.2}$$

since

$$|R_1|^2 + (\mu\sin 2\theta_2/\sin 2\theta_1)|T_1|^2 = 1 \tag{5.3}$$

in this case.

Table 1 compares the values of $|R_1|$ and $|T_1|$ computed from the velocity and potential approximations with the exact values given by (5.1) and (5.2). It can be seen that there is good agreement for all except small μ. Closest agreement occurs near $\mu = 1$, where the exponential forms assumed in the approximations are almost exact and local effects are small.

6. Conclusion

Two approximate methods have been used to determine how an obliquely incident plane wave is affected by a vortex sheet separating two regions of fluid having constant but different velocities. The results are shown to be similar to results obtained by Longuet-Higgins & Stewart (1961), who assumed a gradual change in current velocity. This suggests that in the intermediate range, where the change in current velocity over a wavelength is comparable with the wave velocity and for which no theory has been devised, the curves of the reflexion and transmission coefficients lie between those given in I and those derived here. Confidence in the two approximations is strengthened by a favourable comparison between the approximations for a simpler problem and the known exact result for that problem.

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